Solutions QFT Exam 25/01/16

1. $\mathcal{L} = \frac{1}{2}\partial_\mu \varphi_1 \partial^\mu \varphi_1 - \frac{1}{2} m_1^2 \varphi_1^2 + \frac{1}{2}\partial_\mu \varphi_2 \partial^\mu \varphi_2 - \frac{1}{2} m_2^2 \varphi_2^2 - \frac{\lambda}{2}\varphi_1 \varphi_2^2$

a) Derive the Feynman rule for the vertex.

$\Rightarrow G^{(3)}_{122}$ connected of order λ.

$$G^{(3)}_{122}(x_1, x_2, x_3) = \frac{1}{Z[J_1=J_2=0, \lambda=0]} \int D\varphi_1 D\varphi_2 \, e^{i\int d x^d \mathcal{L}_0 - \frac{\lambda}{2}\varphi_1\varphi_2^2} \varphi_1(x_1)\varphi_2(x_2)\varphi_2(x_3)$$

expand in λ: $= \langle \varphi_1(x_1)\varphi_2(x_2)\varphi_2(x_3)\rangle - \frac{i\lambda}{2}\int d^d w \langle \varphi_1(w)\varphi_2^2(w)\varphi_1(x_1)\varphi_2(x_2)\varphi_2(x_3)\rangle$

$+ \mathcal{O}(\lambda^2)$.

The $\mathcal{O}(\lambda)$ connected contribution is:

$$G^{(3)}_{122c}\Big|_{\mathcal{O}(\lambda)} = \frac{-i\lambda}{2}\int d^d w \; 2 \cdot iD_1(x_1 - w)\, iD_2(x_2 - w)\, iD_3(x_3 - w)$$

$$= -i\lambda \int d^d w \prod_{j=1}^{3} \int \frac{d^d k_j}{(2\pi)^d} e^{ik_1(x_1-w) + ik_2(x_2-w) + ik_3(x_3-w)}$$

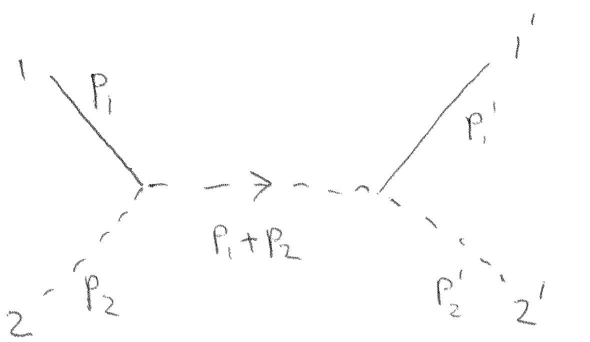
$$\cdot \frac{i}{k_1^2 - m_1^2 + i\varepsilon} \frac{i}{k_2^2 - m_2^2 + i\varepsilon} \frac{i}{k_3^2 - m_2^2 + i\varepsilon}$$

$$= \underset{\uparrow\ \text{vertex rule}}{(-i\lambda)}\ \underbrace{(2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3)}_{\text{conservation of total 4-momentum}} \underbrace{\prod_{j=1}^{3}\int \frac{d^d k_j}{(2\pi)^d} e^{ik_j x_j} \frac{i}{k_1^2 - m_1^2 + i\varepsilon} \cdot \frac{i}{k_2^2 - m_2^2 + i\varepsilon} \cdot \frac{i}{k_3^2 - m_2^2 + i\varepsilon}}_{\text{remove by amputating the external legs}}.$$

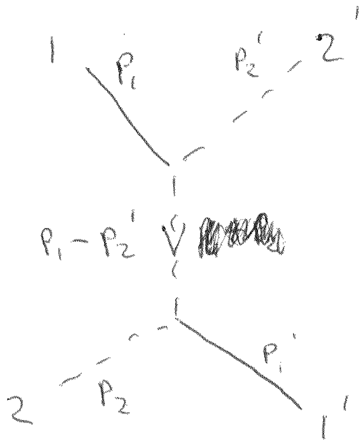
Don't forget to draw the diagram:

(solid line for φ_1,
dashed line for φ_2.)

b) $1\,2 \rightarrow 1\,2$.



s-channel.



u-channel.

$$A_s = (-i\lambda)^2 \frac{i}{s-m_2^2}$$

$$A_u = (-i\lambda)^2 \frac{i}{u-m_2^2}$$

$$A = -i\lambda^2 \left(\frac{1}{s-m_2^2} + \frac{1}{u-m_2^2} \right)$$

Note: $i\varepsilon$ is not needed, because s and u are away from the pole.

There is no t-channel, since this would require a $\varphi_1^2\varphi_2$ and a φ_2^3 vertex, and only the $\varphi_1\varphi_2^2$ interaction is present in this model.

2. $\mathcal{L} = \bar{\Psi}(i\cancel{D} - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + c\bar{\Psi}\sigma^{\mu\nu}\Psi F_{\mu\nu}$.

a) Chiral symmetry

$$\bar{\Psi}' i\gamma^\mu D'_\mu \Psi' = \bar{\Psi} e^{i\alpha\gamma_5} i\gamma^\mu D_\mu e^{i\alpha\gamma_5}\Psi = \bar{\Psi} i\gamma^\mu D_\mu \Psi.$$

since $\{\gamma_5, \gamma_\mu\} = 0$, and $D'_\mu = D_\mu$ since $A'_\mu(x) = A_\mu(x)$.

$$m\bar{\Psi}'\Psi' = m\bar{\Psi}e^{2i\alpha\gamma_5}\Psi \neq m\bar{\Psi}\Psi.$$

$$F'_{\mu\nu}F^{\mu\nu\prime} = F_{\mu\nu}F^{\mu\nu} \quad \text{since } A'_\mu = A_\mu.$$

$$\bar{\Psi}'\sigma^{\mu\nu}\Psi' F'_{\mu\nu} = \bar{\Psi}e^{i\alpha\gamma_5}\sigma^{\mu\nu}e^{i\alpha\gamma_5}\Psi F_{\mu\nu}$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \Rightarrow \{\sigma^{\mu\nu}, \gamma_5\} = \frac{i}{2}\{[\gamma^\mu,\gamma^\nu],\gamma_5\}$$

$$= 2\gamma_5\sigma^{\mu\nu} \quad \text{since } \{\gamma^\mu, \gamma_5\} = 0.$$

$$\Rightarrow e^{i\alpha\gamma_5}\sigma^{\mu\nu}e^{i\alpha\gamma_5} = (1 + i\alpha\gamma_5 + \mathcal{O}(\alpha^2))\sigma^{\mu\nu}(1 + i\alpha\gamma_5 + \mathcal{O}(\alpha^2))$$

$$= \sigma^{\mu\nu} + i\alpha\{\gamma_5, \sigma^{\mu\nu}\} + \mathcal{O}(\alpha^2)$$

$$= \sigma^{\mu\nu} + 2i\alpha\gamma_5\sigma^{\mu\nu} + \mathcal{O}(\alpha^2)$$

$$= e^{2i\alpha\gamma_5}\sigma^{\mu\nu}$$

$$\Rightarrow \bar{\Psi}'\sigma^{\mu\nu}\Psi' F'_{\mu\nu} = e^{2i\alpha\gamma_5}\bar{\Psi}\sigma^{\mu\nu}\Psi F_{\mu\nu} \neq \bar{\Psi}\sigma^{\mu\nu}\Psi F_{\mu\nu}.$$

b) Canonical dimension of c.

$$[S] = 0, \quad [\mathcal{L}] = d, \quad [A_\mu] = \frac{d-2}{2}, \quad [\psi] = \frac{d-1}{2}$$

$\uparrow$

to see this, note that $[F_{\mu\nu}F^{\mu\nu}] = d$

$$\Rightarrow [F_{\mu\nu}] = \frac{d}{2}$$

$$F_{\mu\nu} \sim \partial_\mu A_\nu \Rightarrow [A_\mu] = \frac{d}{2} - 1 = \frac{d-2}{2}.$$

Pauli term; $d = [c] + 2[\psi] + [F_{\mu\nu}]$

$$\Rightarrow [c] = d - 2[\psi] - [F_{\mu\nu}]$$

$$= d - 2\left(\frac{d-1}{2}\right) - \frac{d}{2}$$

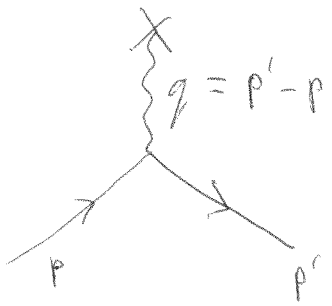
$$= -\frac{d}{2} + 1 = \frac{2-d}{2}.$$

c) $[c] > 0$ superrenormalizable $d < 2$.

$[c] = 0$ renormalizable $d = 2$

$[c] < 0$ non renormalizable $d > 0$.

4.



$q_0 = 0$

$E = E'$

$|\vec{p}| = |\vec{p}'|$

$$A^\mu(\vec{q}) = \left(\frac{Ze}{|\vec{q}|^2}, \ 0, \ 0, \ 0 \right)$$

a)

$$A = \bar{u}_{r'}(\vec{p}')\,(ie\gamma^\mu)\,u_r(\vec{p})\,A_\mu(\vec{q})$$

$$= \frac{iZe^2}{|\vec{q}|^2}\,\bar{u}_{r'}(\vec{p}')\,\gamma^0\,u_r(\vec{p})$$

b) Start by calculating $X = \frac{1}{2}\sum_{r,r'} AA^\dagger$.

$$X = \frac{1}{2}\frac{Z^2e^4}{|\vec{q}|^4}\sum_{r,r'} \bar{u}_{r'}(\vec{p}')\gamma^0 u_r(\vec{p})\,\bar{u}_r(\vec{p})\gamma^0 u_{r'}(\vec{p}')$$

$$= \frac{1}{2}\frac{Z^2e^4}{|\vec{q}|^4}\,\mathrm{Tr}\left[\frac{\not{p}'+m}{2m}\gamma^0\,\frac{\not{p}+m}{2m}\gamma^0\right]$$

since $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$

$$= \frac{Z^2e^4}{2|\vec{q}|^4}\,\frac{1}{4m^2}\cdot 4\left[2p'^0p^0 - p'\cdot p + m^2\right]$$

$$= \frac{Z^2e^4}{2m^2|\vec{q}|^4}\left(p'_0p_0 + \vec{p}'\cdot\vec{p} + m^2\right).$$

Now, we use some relativistic kinematics:

$$E' = E, \quad |\vec{p}'| = |\vec{p}|, \quad p^2 = m^2 \Rightarrow E = \sqrt{\vec{p}^2 + m^2} = \sqrt{\vec{p}'^2 + m^2}.$$

$$\vec{p}' \cdot \vec{p} = |\vec{p}|^2 \cos\theta$$

$$|\vec{q}|^2 = |\vec{p}' - \vec{p}|^2 = 2|\vec{p}|^2 - 2|\vec{p}|^2 \cos\theta$$

$$= 2|\vec{p}|^2(1 - \cos\theta)$$

Now use: $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

In particular: $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$.

$$\Rightarrow 1 - \cos\theta = 1 - \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 2\sin^2\frac{\theta}{2}.$$

$$\Rightarrow |\vec{q}|^2 = 4|\vec{p}|^2 \sin^2\frac{\theta}{2}$$

We now have for X:

$$X = \frac{Z^2 e^4}{2m^2 \, 16 \, |\vec{p}|^4 \sin^4\frac{\theta}{2}} \cdot \left(E^2 + |\vec{p}|^2 \cos\theta + m^2\right)$$

use $\frac{\vec{p}}{E} = \vec{v}$ $\left(m^2 = E^2(1 - v^2)\right)$ so: $|\vec{p}| = v \cdot E$

$$\Rightarrow X = \frac{Z^2 e^4}{32 m^2 E^4 v^4 \sin^4\frac{\theta}{2}} \left(E^2 + v^2 E^2 \cos\theta + m^2\right).$$

Also writing the last m^2 as $E^2(1-v^2)$:

$$X = \frac{Z^2 e^4}{32\, m^2 E^4 v^4 \sin^4\frac{\theta}{2}} \left(E^2 + v^2 E^2 \cos\theta + E^2 - E^2 v^2\right)$$

$$= \quad '' \quad \left(2E^2 - E^2 v^2 (1-\cos\theta)\right)$$

$$= \quad '' \quad 2E^2\left(1 - \cancel{\tfrac{1}{2}} v^2 \cdot \cancel{2} \sin^2\frac{\theta}{2}\right)$$

$$= \quad '' \quad 2E^2\left(1 - v^2 \sin^2\frac{\theta}{2}\right)$$

Now plugging everything into the expression for the unpolarized differential cross-section:

$$\frac{d\sigma}{d\Omega} = \frac{\cancel{m^2}}{4\pi^2} \frac{Z^2 e^4}{32\, \cancel{m^2} E^4 v^4 \sin^4\frac{\theta}{2}} \, 2E^2\left(1 - v^2 \sin^2\frac{\theta}{2}\right)$$

$$= \frac{Z^2 e^4}{4\cdot(4\pi)^2 E^2 v^4 \sin^4\frac{\theta}{2}} \left(1 - v^2 \sin^2\frac{\theta}{2}\right) \qquad \alpha = \frac{e^2}{4\pi}$$

$$= \frac{Z^2 \alpha^2}{4E^2 v^4 \sin^4\frac{\theta}{2}} \left(1 - v^2 \sin^2\frac{\theta}{2}\right).$$